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THE THEORY OF AXISYMMETRIC TURBULENCE

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The present paper completes the theory of axisymmetric tensors and forms to the extent that is needed for the development of a theory of turbulence in which symmetry about a certain preferred direction is assumed to exist. Particular attention is given to the manner in which tensors, solenoidal in one or more indices, can be derived, uniquely, in a gauge-invariant way, as the curl of a suitably defined skew tensor.

The explicit representation of the fundamental velocity correlation tensor $(\overline{u_i u_j'})$ in terms of two defining scalars is found; and the differential equations governing these scalars is also derived. In the theory of axisymmetric turbulence these latter equations replace the equation of von Kármán & Howarth in the theory of isotropic turbulence.

1. Introduction

The statistical theory of isotropic turbulence initiated by Taylor (1935, 1938) has dominated all recent developments in the subject. The central idea in these developments is that of isotropy, which requires the time average of any function of the velocity components, defined with respect to a particular set of axes, to be invariant under arbitrary rotations and reflexions of the axes of reference. A phenomenological theory of turbulence incorporating this idea of isotropy divides itself into two parts: a *kinematical* part which consists in setting up correlations between velocity components (and/or their derivatives) at two different points of the medium and in reducing the form of the associated tensor to meet the requirements of isotropy; and a *dynamical* part which consists in deriving the consequences of the equations of motion and continuity for the fundamental scalar functions defining the correlation tensors. The principal equations of this theory were derived by von Kármán & Howarth (1938); but the most concise and elegant treatment of the subject is due to Robertson (1940), who developed for this purpose a theory of *isotropic tensors*.

While the theory of isotropic turbulence introduces the subject in its simplest context, it is, nevertheless, almost invariably true that whenever turbulence is present, there is also present a preferred direction defined by the direction of the mean flow. Indeed, since fluctuations in the velocity field are generally defined with respect to the local mean values, it would seem that a more natural starting point for the theory will be provided by the concept of *axisymmetry* which will require the mean value of any function of the velocities and their derivatives to be invariant, *not* for the full rotation group, but only for rotations about the preferred direction, $\boldsymbol{\lambda}$ (say), and for reflexions in planes containing $\boldsymbol{\lambda}$ and perpendicular to $\boldsymbol{\lambda}$. The development of such a theory of *axisymmetric turbulence* may be useful in establishing the circumstances under which isotropy may be expected to prevail; and it may also be useful in explaining the emergence of anisotropy during the last stages in the decay of, what was apparently once, isotropic turbulence (Batchelor, private communication).

A beginning in the theory of axisymmetric turbulence was made by Batchelor (1946). However, Batchelor did not develop the theory of *axisymmetric tensors* and *forms* far enough to derive the basic equations of the problem.

In this paper the theory of axisymmetric tensors will be developed to the same degree of completeness that Robertson developed the theory of isotropic tensors. It turns out that the essential part of the theory is that which pertains to solenoidal tensors and their representation as the curl of certain basic *skew* tensors—an aspect of the theory which Batchelor did not consider. With the theory of axisymmetric tensors and forms completed, the reduction of the equations of motion is straightforward. In this manner a pair of equations will be derived which, under the conditions of axisymmetric turbulence, replace the well-known equation of von Kármán & Howarth in the theory of isotropic turbulence.

2. Axisymmetric tensors and forms

Let $F_{ijk\ldots}$ denote a Cartesian tensor; further, let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, etc., denote arbitrary unit vectors. Consider the scalar product

$$F(\mathbf{a},\mathbf{b},\mathbf{c},\ldots) = F_{ijk\ldots}\,a_i b_j c_k\ldots, \tag{1}$$

where here, and in the sequel, summation over the repeated indices is to be understood. Following Batchelor (1946) we shall say that the tensor $F_{ijk\ldots}$ is axially symmetric about a direction specified by a unit vector $\boldsymbol{\lambda}$ if $F(\mathbf{a},\mathbf{b},\mathbf{c},\ldots)$ is invariant for arbitrary rotations about the direction $\boldsymbol{\lambda}$ and for reflexions in planes containing $\boldsymbol{\lambda}$ and normal to $\boldsymbol{\lambda}$. When this is the case we shall say that $F(\mathbf{a},\mathbf{b},\mathbf{c},\ldots)$ is an *axisymmetric form* and $F_{ijk\ldots}$ is an *axisymmetric tensor*.

Before we go any further, it is of interest to see the relation between axisymmetric tensors and forms and isotropic tensors and forms. In the theory of isotropic tensors, the form F defined as in equation (1) is invariant for rotations and reflexions of the configuration formed by the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, etc., and $\boldsymbol{\xi}$, where $\xi_i = x_i'-x_i$ and x_i' and x_i are the co-ordinates of two points at which the correlations of the field variables (velocity, derivatives of velocity, pressure, etc.) are considered. On the other hand, in the theory of axisymmetric tensors, the form F must be invariant for all rotations and reflexions of the vector configuration formed by $\boldsymbol{\xi}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, etc., *and* $\boldsymbol{\lambda}$. To emphasize this dependence of F, in the axisymmetric case, on both $\boldsymbol{\xi}$ and $\boldsymbol{\lambda}$, we shall write $F(\boldsymbol{\xi},\boldsymbol{\lambda};\mathbf{a},\mathbf{b},\mathbf{c},\ldots)$ instead of simply $F(\mathbf{a},\mathbf{b},\mathbf{c},\ldots)$.

The general problem in the theory of axisymmetric tensors is, therefore, to determine the form $F(\boldsymbol{\xi},\boldsymbol{\lambda};\mathbf{a},\mathbf{b},\mathbf{c},\ldots)$ which will be invariant under the full rotation group of the vector configuration formed by $\boldsymbol{\xi}$, $\boldsymbol{\lambda}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, etc. This problem is readily solved, since it follows from the theory of invariants (to which Robertson first directed attention) that $F(\boldsymbol{\xi},\boldsymbol{\lambda};\mathbf{a},\mathbf{b},\mathbf{c},\ldots)$ must be expressible in terms of the *fundamental invariants* satisfying the same conditions, namely, the scalar products of any two of the vectors $\boldsymbol{\xi}$, $\boldsymbol{\lambda}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, etc. (including the scalar squares). Consequently, choosing the combinations of scalar products which will be in conformity with (1), we can write down the most general expression for $F(\boldsymbol{\xi},\boldsymbol{\lambda};\mathbf{a},\mathbf{b},\mathbf{c},\ldots)$.

In addition to axisymmetric tensors and forms, we shall also have to deal with axisymmetric *skew tensors* and forms. The skew tensors—or 'tensor densities' as they are sometimes called—transform as ordinary tensors under proper rotations, but they take the opposite sign to true tensors on reflexion in the origin. It is again not difficult to write down the general expressions for such skew tensors, since the corresponding skew forms must be expressible as sums of products of an odd number of determinants such as $[\boldsymbol{\xi}\mathbf{ab}]$, $[\boldsymbol{\lambda}\mathbf{ab}]$, $[\boldsymbol{\xi}\boldsymbol{\lambda}\mathbf{a}]$, $[\mathbf{abc}]$, etc., formed by any three of the vectors $\boldsymbol{\xi}$, $\boldsymbol{\lambda}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, etc.; and, again, choosing combinations of the scalar products and an odd number of the available determinants which will be in conformity with (1), we can write down the corresponding skew forms and tensors.

3. Linear forms; axisymmetric skew axisymmetric and solenoidal vectors

Let
$$L(\boldsymbol{\xi},\boldsymbol{\lambda};\mathbf{a}) = L_i a_i \tag{2}$$
be the linear form. The scalar products available for its construction are $(\boldsymbol{\xi}.\boldsymbol{\xi})$, $(\boldsymbol{\lambda}.\boldsymbol{\lambda})$, $(\mathbf{a}.\mathbf{a})$, $(\boldsymbol{\xi}.\boldsymbol{\lambda})$, $(\boldsymbol{\lambda}.\mathbf{a})$ and $(\mathbf{a}.\boldsymbol{\xi})$. Hence (cf. Batchelor 1946, equations (2·3) and (2·4))
$$L(\boldsymbol{\xi},\boldsymbol{\lambda};\mathbf{a}) = M(\boldsymbol{\xi}.\mathbf{a})+N(\boldsymbol{\lambda}.\mathbf{a}), \tag{3}$$
where M and N are arbitrary functions of
$$r^2 = (\boldsymbol{\xi}.\boldsymbol{\xi}) \quad \text{and} \quad (\boldsymbol{\xi}.\boldsymbol{\lambda}) = r\mu \quad \text{(say)}. \tag{4}$$
An axisymmetric vector is, therefore, of the form
$$L_i = M\xi_i + N\lambda_i. \tag{5}$$

We shall often have occasion to differentiate expressions such as (5) with respect to ξ_j. For this purpose it is convenient to introduce the differential operators
$$D_r = \frac{1}{r}\frac{\partial}{\partial r} - \frac{\mu}{r^2}\frac{\partial}{\partial\mu} \quad \text{and} \quad D_\mu = \frac{1}{r}\frac{\partial}{\partial\mu}, \tag{6}$$
so that
$$\frac{\partial}{\partial\xi_j} = \xi_j D_r + \lambda_j D_\mu. \tag{7}$$
It is readily verified that the operators D_r and D_μ permute and that
$$D_{r\mu} = D_r D_\mu = D_\mu D_r = \frac{1}{r^2}\frac{\partial^2}{\partial r\,\partial\mu} - \frac{\mu}{r^3}\frac{\partial^2}{\partial\mu^2} - \frac{1}{r^3}\frac{\partial}{\partial\mu}. \tag{8}$$
A further property of the operator D_r is that it permutes with any function of $r\mu$; for
$$D_r f(r\mu) \equiv 0. \tag{9}$$

Now by differentiating (5) with respect to ξ_j, we obtain the second-order axisymmetric tensor
$$\frac{\partial L_i}{\partial \xi_j} = D_r M\xi_i\xi_j + M\delta_{ij} + D_\mu N\lambda_i\lambda_j + D_\mu M\xi_i\lambda_j + D_r N\lambda_i\xi_j. \tag{10}$$
The divergence of the vector L_i is therefore given by
$$\frac{\partial L_i}{\partial \xi_i} = (r^2 D_r + r\mu D_\mu + 3)\, M + (r\mu D_r + D_\mu)\, N. \tag{11}$$

The differential operators
$$r^2 D_r + r\mu D_\mu + n = r\frac{\partial}{\partial r} + n, \tag{12}$$
and
$$r\mu D_r + D_\mu = \mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}, \tag{13}$$

which occur in the expression for the divergence of L_i, are of frequent occurrence in the theory. They satisfy the identity

$$(r^2D_r + r\mu D_\mu + n + 1)\,(r\mu D_r + D_\mu) = (r\mu D_r + D_\mu)\,(r^2D_r + r\mu D\mu + n). \tag{14}$$

Also, if any function $f(r, \mu)$ satisfies the equation

$$(r^2D_r + r\mu D_\mu + n)\,f = r\frac{\partial f}{\partial r} + nf = 0 \quad (n > 0), \tag{15}$$

then

$$fr^{-n} \quad \text{is a function of } \mu \text{ only.} \tag{16}$$

Consequently, if f is assumed to be continuous and bounded in r, then

$$f(r, \mu) \equiv 0. \tag{17}$$

Similarly, if a function $g(r, \mu)$ satisfies the equation

$$(r\mu D_r + D_\mu)\,g = \mu\frac{\partial g}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial g}{\partial \mu} = 0, \tag{18}$$

then

$$g(r, \mu) \equiv g(r\sqrt{(1-\mu^2)}). \tag{19}$$

Returning to equation (11), we observe that the condition that L_i is solenoidal is

$$(r^2D_r + r\mu D_\mu + 3)\,M + (r\mu D_r + D_\mu)\,N = 0. \tag{20}$$

From the identity (14) it now follows that

$$M = -(r\mu D_r + D_\mu)\,L,$$

and

$$N = +(r^2D_r + r\mu D_\mu + 2)\,L, \tag{21}$$

where L is an arbitrary function of r and $r\mu$, satisfy the condition (20). *This representation of a solenoidal axisymmetric vector in terms of a single defining scalar L is unique*, since from $L_i = 0$ (i.e. $M = N = 0$) it follows that $L \equiv 0$ (cf. equations (15) to (19)).

Turning next to the consideration of skew axisymmetric vectors, we observe that the most general skew axisymmetric form is given by

$$l_i a_i = L\epsilon_{ilm} a_i \lambda_l \xi_m, \tag{22}$$

since $[\mathbf{a}\boldsymbol{\lambda}\boldsymbol{\xi}]$ is the only determinant available for its construction. In (22), L is an arbitrary function of r and $r\mu$ and ϵ_{ijk} is the usual alternating tensor which is different from zero only when all three of its indices are different from one another and is equal to $+1$ or -1, according as (ijk) is an even or an odd permutation of (123).

According to (22), a skew axisymmetric vector is defined in terms of a single scalar and is of the form

$$l_i = L\epsilon_{ilm}\lambda_l \xi_m. \tag{23}$$

The derivative of this vector with respect to ξ_j leads to an axisymmetric skew tensor of the second order:

$$\frac{\partial l_i}{\partial \xi_j} = D_r L \xi_j \epsilon_{ilm}\lambda_l \xi_m + D_\mu L \lambda_j \epsilon_{ilm}\lambda_l \xi_m - L\epsilon_{ijm}\lambda_m. \tag{24}$$

The curl of a skew axisymmetric vector is an axisymmetric vector, and since it will be automatically solenoidal we have here a simple method of deriving the representation of a solenoidal axisymmetric vector. Thus, taking the curl of l_i, we have

$$L_i = \epsilon_{ijk}\frac{\partial l_k}{\partial \xi_j} = \epsilon_{ijk}\epsilon_{kmn}\frac{\partial}{\partial \xi_j}L\lambda_m\xi_n$$
$$= (\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})\,[D_r L\lambda_m\xi_n\xi_j + D_\mu L\lambda_m\xi_n\lambda_j + L\lambda_m\delta_{nj}]. \tag{25}$$

Hence
$$L_i = -(r\mu D_r + D_\mu)\,L\xi_i + (r^2 D_r + r\mu D_\mu + 2)\,L\lambda_i. \tag{26}$$

We have thus recovered the representation (21) of a solenoidal axisymmetric vector.

We shall now consider the effect of the Laplacian operator, ∇^2, on an axisymmetric vector. Since ∇^2 is a scalar operator, it is evident that $\nabla^2 L_i$ will be axisymmetric if L_i is axisymmetric. And if L_i is solenoidal, $\nabla^2 L_i$ will also be solenoidal, and in this case it is of interest to inquire into its defining scalar. Since

$$\nabla^2 L_i = \nabla^2\epsilon_{ijk}\frac{\partial l_k}{\partial \xi_j} = \nabla^2\epsilon_{ijk}\frac{\partial}{\partial \xi_j}L\epsilon_{klm}\lambda_l\xi_m = \epsilon_{ijk}\frac{\partial}{\partial \xi_j}\nabla^2 L\epsilon_{klm}\lambda_l\xi_m, \tag{27}$$

it follows that the defining scalar of $\nabla^2 L_i$ is the same as the scalar defining the skew vector obtained by the operation of the Laplacian on l_i itself. We have (cf. equation (24))

$$\nabla^2 l_i = \frac{\partial}{\partial \xi_j}(D_r L\xi_j\epsilon_{ilm}\lambda_l\xi_m + D_\mu L\lambda_j\epsilon_{ilm}\lambda_l\xi_m - L\epsilon_{ijm}\lambda_m). \tag{28}$$

On evaluating the quantity on the right-hand side, we find that we are left with

$$\nabla^2 l_i = (r^2 D_{rr} + 2r\mu D_{r\mu} + D_{\mu\mu} + 5D_r)\,L\epsilon_{ijk}\lambda_j\xi_k. \tag{29*}$$

The defining scalar of $\nabla^2 l_i$ is, therefore,

$$(r^2 D_{rr} + 2r\mu D_{r\mu} + D_{\mu\mu} + 5D_r)\,L, \tag{30}$$

and according to our earlier remarks, this is also the defining scalar of $\nabla^2 L_i$.

Introducing the operator

$$\Delta = r^2 D_{rr} + 2r\mu D_{r\mu} + D_{\mu\mu} + 5D_r$$
$$= \frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2} - \frac{4\mu}{r^2}\frac{\partial}{\partial \mu}, \tag{31}$$

we observe that the operation of ∇^2 on L_i is equivalent to the operation of Δ on its defining scalar. It is of interest to notice that this operator Δ is the same as the one which defines the ultraspherical harmonics in five dimensions (cf. A. Sommerfeld 1949).

4. Bilinear forms; tensors of the second order

Let
$$Q(\boldsymbol{\xi},\boldsymbol{\lambda};\,\mathbf{a},\mathbf{b}) = Q_{ij}a_i b_j \tag{32}$$

denote the required axisymmetric bilinear form. The consideration of the available scalar products shows that (cf. Batchelor 1946)

$$Q(\boldsymbol{\xi},\boldsymbol{\lambda};\,\mathbf{a},\mathbf{b}) = A(\boldsymbol{\xi}.\mathbf{a})\,(\boldsymbol{\xi}.\mathbf{b}) + B(\mathbf{a}.\mathbf{b}) + C(\boldsymbol{\lambda}.\mathbf{a})\,(\boldsymbol{\lambda}.\mathbf{b}) + D(\boldsymbol{\lambda}.\mathbf{a})\,(\boldsymbol{\xi}.\mathbf{b}) + E(\boldsymbol{\xi}.\mathbf{a})\,(\boldsymbol{\lambda}.\mathbf{b}), \tag{33}$$

* $D_{rr} = D_r D_r$ and $D_{\mu\mu} = D_\mu D_\mu$.

where A, B, C, D and E are arbitrary functions of r and $r\mu$. The second-order axisymmetric tensor is therefore given by

$$Q_{ij} = A\xi_i\xi_j + B\delta_{ij} + C\lambda_i\lambda_j + D\lambda_i\xi_j + E\xi_i\lambda_j, \tag{34}$$

and requires five scalar functions for its definition.

If the tensor Q_{ij} is solenoidal with respect to its *second index* j, then (cf. Batchelor 1946)

$$\frac{\partial Q_{ij}}{\partial \xi_j} = \xi_i[(r^2D_r + r\mu D_\mu + 4)\,A + D_r B + (r\mu D_r + D_\mu)\,E] + \lambda_i[D_\mu B + (r\mu D_r + D_\mu)\,C + (r^2 D_r + r\mu D_\mu + 3)\,D + E] = 0. \tag{35}$$

We must accordingly have

$$(r^2D_r + r\mu D_\mu + 4)\,A + D_r B + (r\mu D_r + D_\mu)\,E = 0,$$

and

$$D_\mu B + (r\mu D_r + D_\mu)\,C + (r^2D_r + r\mu D_\mu + 3)\,D + E = 0. \tag{36}$$

From this it would follow that an axisymmetric tensor of the second order, solenoidal in one of its indices, requires three defining scalars. The explicit representation of such a solenoidal tensor, in terms of three defining scalars, is most easily achieved by expressing Q_{ij} as the curl of a suitably defined skew tensor.

Now to construct a skew bilinear form we have the determinants $[\mathbf{a}\mathbf{b}\boldsymbol{\xi}]$, $[\mathbf{a}\mathbf{b}\boldsymbol{\lambda}]$, $[\mathbf{a}\boldsymbol{\lambda}\boldsymbol{\xi}]$ and $[\mathbf{b}\boldsymbol{\lambda}\boldsymbol{\xi}]$ available. From this it would appear that a general skew bilinear form must be a linear combination of

$$[\mathbf{a}\mathbf{b}\boldsymbol{\xi}], \quad [\mathbf{a}\mathbf{b}\boldsymbol{\lambda}], \quad [\mathbf{a}\boldsymbol{\lambda}\boldsymbol{\xi}]\,(\mathbf{b}\,.\,\boldsymbol{\lambda}), \quad [\mathbf{a}\boldsymbol{\lambda}\boldsymbol{\xi}]\,(\mathbf{b}\,.\,\boldsymbol{\xi}), \quad [\mathbf{b}\boldsymbol{\lambda}\boldsymbol{\xi}]\,(\mathbf{a}\,.\,\boldsymbol{\lambda}) \quad \text{and} \quad [\mathbf{b}\boldsymbol{\lambda}\boldsymbol{\xi}]\,(\mathbf{a}\,.\,\boldsymbol{\xi}), \tag{37}$$

with coefficients which are arbitrary functions of r and $r\mu$. Correspondingly, it would appear that the general axisymmetric skew tensor of the second order must be a linear combination of the *six* tensors

$$\epsilon_{ijk}\xi_k, \quad \epsilon_{ijk}\lambda_k, \quad \lambda_j\epsilon_{ilm}\lambda_l\xi_m, \quad \xi_j\epsilon_{ilm}\lambda_l\xi_m, \quad \lambda_i\epsilon_{jlm}\lambda_l\xi_m \quad \text{and} \quad \xi_i\epsilon_{jlm}\lambda_l\xi_m. \tag{38}$$

However, in virtue of the identities

$$\left.\begin{aligned} \xi_j\epsilon_{ilm}\lambda_l\xi_m - \xi_i\epsilon_{jlm}\lambda_l\xi_m &= r\mu\epsilon_{ijk}\xi_k - r^2\epsilon_{ijk}\lambda_k \\ \text{and} \qquad \lambda_j\epsilon_{ilm}\lambda_l\xi_m - \lambda_i\epsilon_{jlm}\lambda_l\xi_m &= \epsilon_{ijk}\xi_k - r\mu\epsilon_{ijk}\lambda_k, \end{aligned}\right\} \tag{39}$$

only four of the six tensors (38) are linearly independent. Also, since (cf. equation (24))

$$Q\epsilon_{ijk}\lambda_k = D_r Q\xi_j\epsilon_{ilm}\lambda_l\xi_m + D_\mu Q\lambda_j\epsilon_{ilm}\lambda_l\xi_m - \frac{\partial l_i}{\partial \xi_j}, \tag{40}$$

where Q is an arbitrary function of r and $r\mu$, it follows that *for purposes of defining a solenoidal axisymmetric tensor according to*

$$Q_{ij} = \epsilon_{jlm}\frac{\partial q_{im}}{\partial \xi_l}, \tag{41}$$

the tensor $Q\epsilon_{ijk}\lambda_k$ *is equivalent to the tensor*

$$D_r Q\xi_j\epsilon_{ilm}\lambda_l\xi_m + D_\mu Q\lambda_j\epsilon_{ilm}\lambda_l\xi_m; \tag{42}$$

for the two tensors differ only by a gradient of a vector, and the curl of this difference, taken in the fashion (41), is zero.* Accordingly, for deriving (in a gauge-invariant way) an axisymmetric tensor Q_{ij}, solenoidal in j, the most general skew form we need consider is

$$q(\boldsymbol{\xi},\boldsymbol{\lambda};\,\mathbf{a},\mathbf{b}) = Q_1[\mathbf{a}\mathbf{b}\boldsymbol{\xi}] + Q_2[\mathbf{a}\boldsymbol{\lambda}\boldsymbol{\xi}]\,(\mathbf{b}\,.\,\boldsymbol{\lambda}) + Q_3[\mathbf{a}\boldsymbol{\lambda}\boldsymbol{\xi}]\,(\mathbf{b}\,.\,\boldsymbol{\xi}), \tag{43}$$

* Essentially, what we are trying to do here is to define the 'potentials' in a gauge-invariant way.

where Q_1, Q_2 and Q_3 are arbitrary functions of r and $r\mu$. The corresponding skew tensor is

$$q_{ij} = Q_1 \epsilon_{ijk} \xi_k + Q_2 \lambda_j \epsilon_{ilm} \lambda_l \xi_m + Q_3 \xi_j \epsilon_{ilm} \lambda_l \xi_m. \tag{44}$$

The curl of this tensor may be readily found; but it is worth noticing that in the reductions use must be made of the identity

$$\epsilon_{ilm} \lambda_l \xi_m \epsilon_{jkn} \lambda_k \xi_n = -r^2 \lambda_i \lambda_j - \xi_i \xi_j + r\mu(\lambda_i \xi_j + \xi_i \lambda_j) + r^2(1-\mu^2)\,\delta_{ij}. \tag{45}$$

In this manner, we find the following explicit representation of an axisymmetric tensor Q_{ij}, solenoidal in j:

$$\begin{aligned} Q_{ij} = {} & [\xi_i \xi_j D_r - \delta_{ij}(r^2 D_r + r\mu D_\mu + 2) + \lambda_i \xi_j D_\mu]\, Q_1 \\ & + [\xi_i \xi_j D_r - \delta_{ij}\{r^2(1-\mu^2)\, D_r + 1\} + \lambda_i \lambda_j (r^2 D_r + 1) - (\lambda_i \xi_j + \xi_i \lambda_j)\, r\mu D_r]\, Q_2 \\ & + [-\xi_i \xi_j D_\mu + \delta_{ij}\{r^2(1-\mu^2)\, D_\mu - r\mu\} - \lambda_i \lambda_j r^2 D_\mu + (\lambda_i \xi_j + \xi_i \lambda_j)\, r\mu D_\mu + \xi_i \lambda_j]\, Q_3. \end{aligned} \tag{46}$$

It can be readily shown that the foregoing representation of Q_{ij} is *unique*; for, from $Q_{ij} = 0$ we can deduce that $Q_1 = Q_2 = Q_3 = 0$* by setting the coefficients of $\xi_i \xi_j$, δ_{ij}, etc., in (46) equal to zero.

Axisymmetric tensors, symmetrical in their indices and solenoidal, play an important part in the theory of turbulence. From equation (46) it now follows that the symmetry of Q_{ij} in its indices requires that

$$Q_3 = D_\mu Q_1 = \frac{1}{r} \frac{\partial Q_1}{\partial \mu}, \tag{47}$$

so that in this case we are left with only two defining scalars. In view of the importance of this result for the theory of turbulence we shall state it in the following form:

A solenoidal axisymmetric tensor

$$Q_{ij} = A \xi_i \xi_j + B \delta_{ij} + C \lambda_i \lambda_j + D(\lambda_i \xi_j + \xi_i \lambda_j), \tag{48}$$

symmetrical in its indices can be derived in a gauge-invariant way from the skew tensor

$$q_{ij} = Q_1 \epsilon_{ijk} \xi_k + Q_2 \lambda_j \epsilon_{ilm} \lambda_l \xi_m + \frac{1}{r} \frac{\partial Q_1}{\partial \mu} \xi_j \epsilon_{ilm} \lambda_l \xi_m, \tag{49}$$

where Q_1 and Q_2 are two arbitrary functions of r and $r\mu$; the coefficients of the tensor expressed in terms of the two defining scalars, Q_1 and Q_2, are (cf. equation (46))

$$\left.\begin{aligned} A &= (D_r - D_{\mu\mu})\, Q_1 + D_r Q_2, \\ B &= [-(r^2 D_r + r\mu D_\mu + 2) + r^2(1-\mu^2)\, D_{\mu\mu} - r\mu D_\mu]\, Q_1 - [r^2(1-\mu^2)\, D_r + 1]\, Q_2, \\ C &= -r^2 D_{\mu\mu} Q_1 + (r^2 D_r + 1)\, Q_2, \\ \text{and} \quad D &= (r\mu D_\mu + 1)\, D_\mu Q_1 - r\mu D_r Q_2. \end{aligned}\right\} \tag{50}$$

Considering next the effect of the Laplacian operator on Q_{ij}, we first observe that $\nabla^2 Q_{ij}$ will be axisymmetric and solenoidal in j, if Q_{ij} is axisymmetric and solenoidal in j. If Q_1, Q_2 and Q_3 are the defining scalars of Q_{ij}, we can find the defining scalars of $\nabla^2 Q_{ij}$ by applying the Laplacian directly on q_{ij} given by equation (44), since the operations of the curl and the

* Provided that none of these functions has singularities at the origin.

Laplacian are interchangeable. Evaluating then the Laplacian of $Q_1\epsilon_{ijk}\xi_k$, $Q_2\lambda_j\epsilon_{ilm}\lambda_l\xi_m$ and $Q_3\xi_j\epsilon_{ilm}\lambda_l\xi_m$, we find

$$\nabla^2 Q_1\epsilon_{ijk}\xi_k = \Delta Q_1\epsilon_{ijk}\xi_k + 2D_\mu Q_1\epsilon_{ijk}\lambda_k, \tag{51}$$

$$\nabla^2 Q_2\lambda_j\epsilon_{ilm}\lambda_l\xi_m = \Delta Q_2\lambda_j\epsilon_{ilm}\lambda_l\xi_m, \tag{52}$$

and $$\nabla^2 Q_3\xi_j\epsilon_{ilm}\lambda_l\xi_m = \Delta Q_3\xi_j\epsilon_{ilm}\lambda_l\xi_m + 2\{D_r Q_3\xi_j\epsilon_{ilm}\lambda_l\xi_m + D_\mu Q_3\lambda_j\epsilon_{ilm}\lambda_l\xi_m - Q_3\epsilon_{ijk}\lambda_k\}, \tag{53}$$

where Δ denotes the differential operator we have defined in equation (31).

To obtain the defining scalars of $\nabla^2 Q_{ij}$ in a gauge-invariant way, we must replace $D_\mu Q_1\epsilon_{ijk}\lambda_k$ in (51) by (cf. equation (42))

$$D_{r\mu}Q_1\xi_j\epsilon_{ilm}\lambda_l\xi_m + D_{\mu\mu}Q_1\lambda_j\epsilon_{ilm}\lambda_l\xi_m. \tag{54}$$

Also, we can neglect the terms in curly brackets in (53), since this is a gradient of a vector (cf. equation (42)) and has a vanishing curl. Combining these results, we have

$$\operatorname{curl}\nabla^2 q_{ij} = \operatorname{curl}\{\Delta Q_1\epsilon_{ijk}\xi_k + (\Delta Q_2 + 2D_{\mu\mu}Q_1)\lambda_j\epsilon_{ilm}\lambda_l\xi_m + (\Delta Q_3 + 2D_{r\mu}Q_1)\xi_j\epsilon_{ilm}\lambda_l\xi_m\}. \tag{55}$$

The defining scalars of $\nabla^2 Q_{ij}$ are, therefore,

$$\Delta Q_1, \quad \Delta Q_2 + 2D_{\mu\mu}Q_1 \quad \text{and} \quad \Delta Q_3 + 2D_{r\mu}Q_1. \tag{56}$$

If Q_{ij} is symmetrical in its indices $Q_3 = D_\mu Q_1$ (cf. equation (47)) and the scalars (56) become

$$\Delta Q_1, \quad \Delta Q_2 + 2D_{\mu\mu}Q_1 \quad \text{and} \quad \Delta D_\mu Q_1 + 2D_{r\mu}Q_1. \tag{57}$$

On the other hand,

$$\begin{aligned} D_\mu \Delta Q_1 &= D_\mu(r^2 D_{rr} + 2r\mu D_{r\mu} + D_{\mu\mu} + 5D_r)\,Q_1 \\ &= (r^2 D_{rr} + 2r\mu D_{r\mu} + D_{\mu\mu} + 5D_r)\,D_\mu Q_1 + 2D_{r\mu}Q_1 \\ &= \Delta D_\mu Q_1 + 2D_{r\mu}Q_1, \end{aligned} \tag{58}$$

since D_μ permutes with all the terms in Δ except $r\mu D_{r\mu}$ and

$$D_\mu(r\mu D_{r\mu}) = r\mu D_{r\mu\mu} + D_{r\mu}. \tag{59}$$

Hence, instead of (57), we may write

$$\Delta Q_1, \quad \Delta Q_2 + 2D_{\mu\mu}Q_1 \quad \text{and} \quad D_\mu(\Delta Q_1). \tag{60}$$

This verifies that $\nabla^2 Q_{ij}$ is symmetrical in its indices. The defining scalars of $\nabla^2 Q_{ij}$, in this case, are therefore

$$\Delta Q_1 \quad \text{and} \quad \Delta Q_2 + 2D_{\mu\mu}Q_1. \tag{61}$$

So far we have considered the operation of curl as applying only to the second index. The operation of taking curl twice on Q_{ij} would therefore mean

$$\begin{aligned} \operatorname{curl}\operatorname{curl} Q_{ij} &= \epsilon_{jlm}\frac{\partial}{\partial \xi_l}\epsilon_{mkn}\frac{\partial}{\partial \xi_k}Q_{in} \\ &= (\delta_{jk}\delta_{nl} - \delta_{jn}\delta_{lk})\frac{\partial^2}{\partial \xi_l\,\partial \xi_k}Q_{in} \\ &= \frac{\partial}{\partial \xi_j}\left(\frac{\partial Q_{in}}{\partial \xi_n}\right) - \frac{\partial^2 Q_{ij}}{\partial \xi_l\,\partial \xi_l} = -\nabla^2 Q_{ij}, \end{aligned} \tag{62}$$

if Q_{ij} is assumed to be solenoidal in j. However, in the theory of turbulence we shall have occasion to consider the effect of taking the curl first with respect to one index and then with respect to the other index, of a tensor Q_{ij} symmetrical in its indices. In other words, we shall have to consider a tensor of the form

$$\Omega_{ij} = \epsilon_{ikn}\frac{\partial}{\partial \xi_k}\epsilon_{jlm}\frac{\partial}{\partial \xi_l}Q_{nm}, \tag{63}$$

where Q_{ij} is symmetrical in its indices. Defined in this manner Ω_{ij} is clearly symmetrical and solenoidal in its indices, and we may ask what its defining scalars are. Since this question has a meaning whether or not Q_{ij} is itself solenoidal, we shall start with the general tensor

$$Q_{ij} = A\xi_i\xi_j + B\delta_{ij} + C\lambda_i\lambda_j + D(\lambda_i\xi_j + \xi_i\lambda_j). \tag{64}$$

Taking the curl of this with respect to the second index, we find

$$q_{ij} = \epsilon_{jlm}\frac{\partial Q_{im}}{\partial \xi_l} = (-A + D_r B)\,\epsilon_{ijk}\xi_k + (D_\mu B - D)\,\epsilon_{ijk}\lambda_k + (D_\mu D - D_r C)\,\lambda_i\epsilon_{jlm}\lambda_l\xi_m + (D_\mu A - D_r D)\,\xi_i\epsilon_{jlm}\lambda_l\xi_m. \tag{65}$$

Now in accordance with (63) we must take the curl of q_{ij} with respect to its first index. On the other hand, since we are interested only in the defining scalars of Ω_{ij}, it should be possible to find them without having to take the curl of (65) explicitly. Indeed, we can argue in the following manner:

First, we observe that taking the curl of $Q\epsilon_{ijk}\xi_k$ with respect to its first index is equivalent to taking the curl of $-Q\epsilon_{ijk}\xi_k$ with respect to its second index provided we interchange the coefficients of $\lambda_i\xi_j$ and $\xi_i\lambda_j$ after taking the curl. The same remark applies also to taking the curl of $Q\epsilon_{ijk}\lambda_k$. Similarly, taking the curl of $Q\xi_i\epsilon_{jlm}\lambda_l\xi_m$ with respect to i is equivalent to taking the curl of $Q\xi_j\epsilon_{ilm}\lambda_l\xi_m$ with respect to j, provided we again interchange the coefficients of $\lambda_i\xi_j$ and $\xi_i\lambda_j$. However, taking the curl of $Q\lambda_i\epsilon_{jlm}\lambda_l\xi_m$ with respect to i is exactly the same as taking the curl of $Q\lambda_j\epsilon_{ilm}\lambda_l\xi_m$ with respect to j, since in this case the resulting tensor is symmetrical in its indices (cf. equation (46)). Now returning to equation (65) we can assert, in view of the foregoing remarks, that

$$\epsilon_{ilm}\frac{\partial q_{mj}}{\partial \xi_l} = \epsilon_{jlm}\frac{\partial q'_{im}}{\partial \xi_l}, \tag{66}$$

where
$$q'_{ij} = (A - D_r B)\,\epsilon_{ijk}\xi_k + (D - D_\mu B)\,\epsilon_{ijk}\lambda_k + (D_\mu D - D_r C)\,\lambda_j\epsilon_{ilm}\lambda_l\xi_m + (D_\mu A - D_r D)\,\xi_j\epsilon_{ilm}\lambda_l\xi_m, \tag{67}$$

since we know that the final result is a symmetrical tensor.

To obtain the defining scalars of Ω_{ij} we must replace $(D - D_\mu B)\,\epsilon_{ijk}\lambda_k$ in (67), in accordance with equation (40), by

$$D_r(D - D_\mu B)\,\xi_j\epsilon_{ilm}\lambda_l\xi_m + D_\mu(D - D_\mu B)\,\lambda_j\epsilon_{ilm}\lambda_l\xi_m. \tag{68}$$

Thus, Ω_{ij} is derived from the skew tensor

$$\omega_{ij} = (A - D_r B)\,\epsilon_{ijk}\xi_k + (2D_\mu D - D_r C - D_{\mu\mu}B)\,\lambda_j\epsilon_{ilm}\lambda_l\xi_m + D_\mu(A - D_r B)\,\xi_j\epsilon_{ilm}\lambda_l\xi_m. \tag{69}$$

The defining scalars of Ω_{ij} are, therefore,

$$\Omega_1 = A - D_r B \quad \text{and} \quad \Omega_2 = 2D_\mu D - D_r C - D_{\mu\mu}B. \tag{70}$$

In practical applications, the case of greatest interest is when Q_{ij} is already solenoidal in its indices. In this case the coefficients A, B, etc., have the values given in equations (50). Inserting these values in (70) we find

$$\left.\begin{aligned} \Omega_1 &= \Delta Q_1 - [r^2(1-\mu^2)\,D_r + 4]\,(D_{\mu\mu}\,Q_1 - D_r\,Q_2), \\ \text{and}\qquad \Omega_2 &= -\Delta Q_2 + D_{\mu\mu}[2(r^2 D_r + 2r\mu D_\mu + 2)\,Q_1 \\ &\qquad - r^2(1-\mu^2)\,(D_{\mu\mu}\,Q_1 - D_r\,Q_2) + 2Q_2]. \end{aligned}\right\} \tag{71}$$

The last problem we shall consider in the theory of second-order tensors is the question of the defining scalars of the tensor (cf. equation (10))

$$L_{ij} = \frac{\partial L_j}{\partial \xi_i} = D_r M \xi_i \xi_j + M\delta_{ij} + D_\mu N \lambda_i \lambda_j + D_\mu M \lambda_i \xi_j + D_r N \xi_i \lambda_j, \tag{72}$$

where L_j is solenoidal so that M and N are of the forms given by equations (21).

First it may be readily verified that

$$\frac{\partial L_j}{\partial \xi_i} = \epsilon_{jlm} \frac{\partial t_{im}}{\partial \xi_l}, \tag{73}$$

where

$$t_{ij} = M\epsilon_{ijk}\xi_k + N\epsilon_{ijk}\lambda_k. \tag{74}$$

For evaluating the curl of t_{ij} in the usual fashion we find that

$$\begin{aligned} \epsilon_{jlm}\frac{\partial t_{im}}{\partial \xi_l} = {} & [\xi_i\xi_j D_r - \delta_{ij}(r^2 D_r + r\mu D_\mu + 2) + \lambda_i \xi_j D_\mu]\,M \\ & + [-\delta_{ij}(r\mu D_r + D_\mu) + \lambda_i\lambda_j D_\mu + \lambda_j \xi_i D_r]\,N. \end{aligned} \tag{75}$$

The agreement of this expression with (72) requires only that

$$M = -(r^2 D_r + r\mu D_\mu + 2)\,M - (r\mu D_r + D_\mu)\,N; \tag{76}$$

and this is true, as may be directly seen by substituting for M and N according to equations (21) and using the identity (14).

Finally, replacing $N\epsilon_{ijk}\lambda_k$ by

$$D_r N \xi_j \epsilon_{ilm} \lambda_l \xi_m + D_\mu N \lambda_j \epsilon_{ilm} \lambda_l \xi_m,$$

we conclude that the defining scalars of $\partial L_j/\partial \xi_i$ are

$$M, \quad D_\mu N \quad \text{and} \quad D_r N. \tag{77}$$

5. Trilinear forms: tensors of the third order

From a consideration of the available scalar products, it can be readily shown that the general third-order axisymmetric tensor must be of the form

$$\begin{aligned} T_{ijk} = {} & F\xi_i\xi_j\xi_k + G\lambda_i\lambda_j\lambda_k \\ & + H^{(1)}\xi_i\xi_j\lambda_k + H^{(2)}\xi_j\xi_k\lambda_i + H^{(3)}\xi_k\xi_i\lambda_j \\ & + I^{(1)}\xi_i\lambda_j\lambda_k + I^{(2)}\xi_j\lambda_k\lambda_i + I^{(3)}\xi_k\lambda_i\lambda_j \\ & + J^{(1)}\xi_i\delta_{jk} + J^{(2)}\xi_j\delta_{ki} + J^{(3)}\xi_k\delta_{ij} \\ & + K^{(1)}\lambda_i\delta_{jk} + K^{(2)}\lambda_j\delta_{ki} + K^{(3)}\lambda_k\delta_{ij}, \end{aligned} \tag{78}$$

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where F, G, etc., are 14 arbitrary functions of r and $r\mu$. If the tensor is solenoidal in one of its indices, then there will be five differential equations between the 14 arbitrary functions; for, the divergence of T_{ijk} will be a tensor of the second order with five components (cf. equation (34)), and the coefficient of each of these must be set equal to zero. Thus, in this case the tensor T_{ijk} will require nine defining scalars.

In the theory of turbulence we shall be mostly concerned with third-order tensors which are symmetrical in two of its indices (say i and j) and solenoidal in the third (say k). This reduces the number of scalars to *six* since the symmetry requires

$$H^{(2)} = H^{(3)}, \quad I^{(1)} = I^{(2)}, \quad J^{(1)} = J^{(2)} \quad \text{and} \quad K^{(1)} = K^{(2)}, \tag{79}$$

and reduces the number of arbitrary functions in (78) to ten, and the condition that the tensor is solenoidal will lead to four* differential equations between them.

The explicit representation of a third-order axisymmetric tensor solenoidal in one of its indices in terms of nine or six defining scalars as the case may be is most easily achieved by deriving it, in a gauge-invariant way, from a suitably defined skew tensor.

Now a consideration of the available determinants shows that there are 29 skew tensors of the third order. They are

$$\epsilon_{ijk}, \quad \epsilon_{ilm}\lambda_l\xi_m\epsilon_{jrs}\lambda_r\xi_s\epsilon_{kpq}\lambda_p\xi_q, \tag{80}$$

$$\left.\begin{matrix} \xi_i\epsilon_{jkl}\xi_l, & \xi_i\epsilon_{jkl}\lambda_l, & \lambda_i\epsilon_{jkl}\lambda_l, & \lambda_i\epsilon_{jkl}\xi_l, \\ \xi_j\epsilon_{kil}\xi_l, & \xi_j\epsilon_{kil}\lambda_l, & \lambda_j\epsilon_{kil}\lambda_l, & \lambda_j\epsilon_{kil}\xi_l, \\ \xi_k\epsilon_{ijl}\xi_l, & \xi_k\epsilon_{ijl}\lambda_l, & \lambda_k\epsilon_{ijl}\lambda_l, & \lambda_k\epsilon_{ijl}\xi_l, \end{matrix}\right\} \tag{81}$$

$$\left.\begin{matrix} \xi_i\xi_j\epsilon_{klm}\lambda_l\xi_m, & \xi_j\xi_k\epsilon_{ilm}\lambda_l\xi_m, & \xi_k\xi_i\epsilon_{jlm}\lambda_l\xi_m, \\ \xi_i\lambda_j\epsilon_{klm}\lambda_l\xi_m, & \xi_j\lambda_k\epsilon_{ilm}\lambda_l\xi_m, & \xi_k\lambda_i\epsilon_{jlm}\lambda_l\xi_m, \\ \lambda_i\lambda_j\epsilon_{klm}\lambda_l\xi_m, & \lambda_j\lambda_k\epsilon_{ilm}\lambda_l\xi_m, & \lambda_k\lambda_i\epsilon_{jlm}\lambda_l\xi_m, \\ \lambda_i\xi_j\epsilon_{klm}\lambda_l\xi_m, & \lambda_j\xi_k\epsilon_{ilm}\lambda_l\xi_m, & \lambda_k\xi_i\epsilon_{jlm}\lambda_l\xi_m, \end{matrix}\right\} \tag{82}$$

$$\delta_{ij}\epsilon_{klm}\lambda_l\xi_m, \quad \delta_{jk}\epsilon_{ilm}\lambda_l\xi_m \quad \text{and} \quad \delta_{ki}\epsilon_{jlm}\lambda_l\xi_m. \tag{83}$$

However, not all of these 29 tensors are linearly independent; only 14 of them are.† The reduction of the 29 tensors to a minimal set of 14 can be accomplished in the following manner:

First, multiplying the identities (39) by ξ_k and λ_k, we obtain the relations

$$\left.\begin{aligned} \xi_j\xi_k\epsilon_{ilm}\lambda_l\xi_m - \xi_i\xi_k\epsilon_{jlm}\lambda_l\xi_m &= r\mu\xi_k\epsilon_{ijl}\xi_l - r^2\xi_k\epsilon_{ijl}\lambda_l, \\ \xi_j\lambda_k\epsilon_{ilm}\lambda_l\xi_m - \xi_i\lambda_k\epsilon_{jlm}\lambda_l\xi_m &= r\mu\lambda_k\epsilon_{ijl}\xi_l - r^2\lambda_k\epsilon_{ijl}\lambda_l, \\ \lambda_j\xi_k\epsilon_{ilm}\lambda_l\xi_m - \lambda_i\xi_k\epsilon_{jlm}\lambda_l\xi_m &= \xi_k\epsilon_{ijl}\xi_l - r\mu\xi_k\epsilon_{ijl}\lambda_l, \\ \lambda_j\lambda_k\epsilon_{ilm}\lambda_l\xi_m - \lambda_i\lambda_k\epsilon_{jlm}\lambda_l\xi_m &= \lambda_k\epsilon_{ijl}\xi_l - r\mu\lambda_k\epsilon_{ijl}\lambda_l. \end{aligned}\right\} \tag{84}$$

These four relations, together with the four additional ones obtained by a cyclical permutation of the indices (ijk), enable us to eliminate eight of the twelve tensors (82). From

* Not five, since the divergence with respect to k of a tensor T_{ijk}, symmetrical in i and j, is a second-order tensor symmetrical in its indices.

† That there are 14 independent skew tensors of the third order can be deduced by starting with the most general *fourth*-order axisymmetric tensor and then contracting it with the alternating tensor (cf. Robertson 1940).

this group of twelve tensors we need to retain only four; and we shall select the four which occur with the factor $\epsilon_{klm}\lambda_l\xi_m$.

Next, the identities

$$\left.\begin{aligned}\xi_i\epsilon_{jkl}\xi_l+\xi_j\epsilon_{kil}\xi_l+\xi_k\epsilon_{ijl}\xi_l &= r^2\epsilon_{ijk},\\ \xi_i\epsilon_{jkl}\lambda_l+\xi_j\epsilon_{kil}\lambda_l+\xi_k\epsilon_{ijl}\lambda_l &= r\mu\epsilon_{ijk},\\ \lambda_i\epsilon_{jkl}\xi_l+\lambda_j\epsilon_{kil}\xi_l+\lambda_k\epsilon_{ijl}\xi_l &= r\mu\epsilon_{ijk},\\ \lambda_i\epsilon_{jkl}\lambda_l+\lambda_j\epsilon_{kil}\lambda_l+\lambda_k\epsilon_{ijl}\lambda_l &= \epsilon_{ijk},\end{aligned}\right\} \tag{85}$$

enable us to eliminate one of the tensors in each of the four groups of three included in (81). Similarly, the identity

$$\xi_i\epsilon_{jkl}\lambda_l-\lambda_i\epsilon_{jkl}\xi_l=\delta_{ij}\epsilon_{klm}\lambda_l\xi_m-\delta_{ik}\epsilon_{jlm}\lambda_l\xi_m, \tag{86}$$

and the two additional ones obtained by cyclically permuting the indices, enable us to eliminate two of the three tensors (83). Finally, the identity (45) enables us to eliminate the second of the two tensors (80). Thus, from the 29 tensors (80) to (83) we obtain the following 14 which are linearly independent:

$$\left.\begin{matrix}\xi_i\xi_j\epsilon_{klm}\lambda_l\xi_m, & \xi_i\epsilon_{jkl}\xi_l, & \lambda_i\epsilon_{jkl}\lambda_l,\\ \xi_i\lambda_j\epsilon_{klm}\lambda_l\xi_m, & \xi_j\epsilon_{kil}\xi_l, & \lambda_j\epsilon_{kil}\lambda_l,\\ \lambda_i\xi_j\epsilon_{klm}\lambda_l\xi_m, & \xi_i\epsilon_{jkl}\lambda_l, & \lambda_i\epsilon_{jkl}\xi_l,\\ \lambda_i\lambda_j\epsilon_{klm}\lambda_l\xi_m, & \xi_j\epsilon_{kil}\lambda_l, & \lambda_j\epsilon_{kil}\xi_l,\\ \delta_{ij}\epsilon_{klm}\lambda_l\xi_m & \text{and} & \epsilon_{ijk}.\end{matrix}\right\} \tag{87}$$

For the purposes of deriving axisymmetric solenoidal tensors in a gauge-invariant way, we can reject some more of the tensors (87) by considering whether the curl of one of them, times an arbitrary function of r and $r\mu$, can be expressed as a linear combination of the others. As to which of the tensors (87) can be rejected on these grounds can be decided by evaluating the gradients of the second-order skew tensors (38) times an arbitrary function of r and $r\mu$. Thus considering the first of the tensors (38), we have

$$\frac{\partial}{\partial\xi_k}Q\epsilon_{ijl}\xi_l=Q\epsilon_{ijk}+D_rQ\xi_k\epsilon_{ijl}\xi_l+D_\mu Q\lambda_k\epsilon_{ijl}\xi_l, \tag{88}$$

where Q is an arbitrary function of r and r_μ. Hence

$$Q\epsilon_{ijk}+D_rQ\xi_k\epsilon_{ijl}\xi_l+D_\mu Q\lambda_k\epsilon_{ijl}\xi_l \tag{89}$$

has a vanishing curl with respect to k. A similar consideration of the other second-order skew tensors shows that the following additional combinations of the third-order tensors are divergence free:

$$\left.\begin{aligned}&Q\lambda_j\epsilon_{kil}\lambda_l+(D_rQ\lambda_j\xi_k+D_\mu Q\lambda_j\lambda_k)\,\epsilon_{ilm}\lambda_l\xi_m,\\ &Q\lambda_i\epsilon_{jkl}\lambda_l-(D_rQ\xi_k\lambda_j+D_\mu Q\lambda_i\lambda_k)\,\epsilon_{jlm}\lambda_l\xi_m,\\ &Q(\xi_j\epsilon_{kil}\lambda_l+\delta_{jk}\epsilon_{ilm}\lambda_l\xi_m)+(D_rQ\xi_k\xi_j+D_\mu Q\lambda_k\xi_j)\,\epsilon_{ilm}\lambda_l\xi_m,\\ &Q(\xi_i\epsilon_{jlk}\lambda_l+\delta_{ik}\epsilon_{jlm}\lambda_l\xi_m)+(D_rQ\xi_i\xi_k+D_\mu Q\xi_i\lambda_k)\,\epsilon_{jlm}\lambda_l\xi_m.\end{aligned}\right\} \tag{90}$$

Accordingly we may eliminate from (87) the following five tensors:

$$\epsilon_{ijk},\quad \xi_j\epsilon_{kil}\lambda_l,\quad \lambda_j\epsilon_{kil}\lambda_l,\quad \xi_i\epsilon_{jkl}\lambda_l\quad\text{and}\quad \lambda_i\epsilon_{jkl}\lambda_l. \tag{91}$$

After all these eliminations, we are left with nine tensors out of the original 29; and the most general skew tensor we need consider is

$$\begin{aligned} t_{ijk} = {} & (T_1\xi_i\xi_j+T_2\lambda_i\lambda_j+T_3\delta_{ij})\,\epsilon_{klm}\lambda_l\xi_m \\ & +T_4\xi_i\epsilon_{jkl}\xi_l+T_5\lambda_i\epsilon_{jkl}\xi_l+T_6\xi_i\lambda_j\epsilon_{klm}\lambda_l\xi_m \\ & +T_7\xi_j\epsilon_{ikl}\xi_l+T_8\lambda_j\epsilon_{ikl}\xi_l+T_9\lambda_i\xi_j\epsilon_{klm}\lambda_l\xi_m, \end{aligned} \tag{92}$$

where $T_1, T_2, \ldots, T_9$ are nine arbitrary functions of r and $r\mu$.

On taking the curl of (92), we obtain the following representation of a third-order axisymmetric tensor T_{ijk}, solenoidal in k, in terms of the nine defining scalars $T_1, \ldots, T_9$:

$$\begin{aligned} T_{ijk} = {} & \epsilon_{klm}\frac{\partial t_{ijm}}{\partial \xi_l} \\ = {} & [-\xi_i\xi_j\xi_k(r\mu D_r+D_\mu)+\xi_i\xi_j\lambda_k(r^2D_r+r\mu D_\mu+4)-\xi_j\xi_k\lambda_i-\xi_k\xi_i\lambda_j]\,T_1 \\ & +[\lambda_i\lambda_j\lambda_k(r^2D_r+r\mu D_\mu+2)-\xi_k\lambda_i\lambda_j(r\mu D_r+D_\mu)]\,T_2 \\ & +[-\delta_{ij}\xi_k(r\mu D_r+D_\mu)+\delta_{ij}\lambda_k(r^2D_r+r\mu D_\mu+2)]\,T_3 \\ & +[\xi_i\xi_j\xi_k D_r+\xi_k\xi_i\lambda_j D_\mu-\xi_i\delta_{jk}(r^2D_r+r\mu D_\mu+3)+\xi_k\delta_{ij}]\,T_4 \\ & +[\xi_i\xi_j\xi_k D_r+\xi_j\xi_k\lambda_i D_\mu-\xi_j\delta_{ik}(r^2D_r+r\mu D_\mu+3)+\xi_k\delta_{ij}]\,T_7 \\ & +[\xi_j\xi_k\lambda_i D_r+\xi_k\lambda_i\lambda_j D_\mu-\lambda_i\delta_{jk}(r^2D_r+r\mu D_\mu+2)]\,T_5 \\ & +[\xi_k\xi_i\lambda_j D_r+\xi_k\lambda_i\lambda_j D_\mu-\lambda_j\delta_{ik}(r^2D_r+r\mu D_\mu+2)]\,T_8 \\ & +[-\xi_k\xi_i\lambda_j(r\mu D_r+D_\mu)+\xi_i\lambda_j\lambda_k(r^2D_r+r\mu D_\mu+3)-\xi_k\lambda_i\lambda_j]\,T_6 \\ & +[-\xi_j\xi_k\lambda_i(r\mu D_r+D_\mu)+\xi_j\lambda_k\lambda_i(r^2D_r+r\mu D_\mu+3)-\xi_k\lambda_i\lambda_j]\,T_9. \end{aligned} \tag{93}$$

If T_{ijk} is in addition symmetrical in i and j, we must put

$$T_4 = T_7, \quad T_5 = T_8 \quad \text{and} \quad T_6 = T_9 \tag{94}$$

in (93). In this manner we find after some rearranging of the terms that

$$\begin{aligned} T_{ijk} = {} & \xi_i\xi_j\xi_k[-(r\mu D_r+D_\mu)\,T_1+2D_rT_4] \\ & +\lambda_i\lambda_j\lambda_k(r^2D_r+r\mu D_\mu+2)\,T_2+\xi_i\xi_j\lambda_k(r^2D_r+r\mu D_\mu+4)\,T_1 \\ & +\xi_k(\xi_i\lambda_j+\lambda_i\xi_j)\,[-T_1+D_\mu T_4+D_rT_5-(r\mu D_r+D_\mu)\,T_6] \\ & +\lambda_k(\xi_i\lambda_j+\lambda_i\xi_j)\,(r^2D_r+r\mu D_\mu+3)\,T_6 \\ & +\xi_k\lambda_i\lambda_j[-(r\mu D_r+D_\mu)\,T_2+2D_\mu T_5-2T_6] \\ & -(\xi_i\delta_{jk}+\xi_j\delta_{ik})\,(r^2D_r+r\mu D_\mu+3)\,T_4 \\ & +\xi_k\delta_{ij}[-(r\mu D_r+D_\mu)\,T_3+2T_4] \\ & -(\lambda_i\delta_{jk}+\lambda_j\delta_{ik})\,(r^2D_r+r\mu D_\mu+2)\,T_5 \\ & +\lambda_k\delta_{ij}(r^2D_r+r\mu D_\mu+2)\,T_3; \end{aligned} \tag{95}$$

this is the explicit representation of T_{ijk} in terms of six defining scalars.

It is of interest to note that it directly follows from (95) that the scalar defining the solenoidal vector T_{iik} is

$$r^2T_1+T_2+3T_3-2T_5+2r\mu T_6. \tag{96}$$

The third-order tensor (95) makes its appearance in the theory of turbulence in the form of the second-order tensor

$$T_{ij} = \frac{\partial}{\partial \xi_k} T_{ikj}. \tag{97}$$

This tensor is solenoidal in j, and its three defining scalars can be found most directly by similarly contracting the skew tensor (i.e. the tensor given by equation (92) with T_7, T_8 and T_9 put equal to T_4, T_5 and T_6, respectively) from which T_{ijk} is derived. Thus

$$T_{ij} = \epsilon_{jlm} \frac{\partial t_{im}}{\partial \xi_l},$$

where

$$t_{ij} = \frac{\partial}{\partial \xi_k} t_{ikj}. \tag{98}$$

On evaluating t_{ij} we find

$$t_{ij} = t_1 \epsilon_{ijk} \xi_k + t_2 \lambda_i \epsilon_{jlm} \lambda_l \xi_m + t_3 \xi_i \epsilon_{jlm} \lambda_l \xi_m + t_4 \epsilon_{ijk} \lambda_k, \tag{99}$$

where

$$\left.\begin{aligned} t_1 &= (r^2 D_r + r\mu D_\mu + 5)\, T_4 + (r\mu D_r + D_\mu)\, T_5, \\ t_2 &= (r\mu D_r + D_\mu)\, T_2 + D_\mu T_3 - D_\mu T_5 + (r^2 D_r + r\mu D_\mu + 5)\, T_6, \\ t_3 &= (r\mu D_r + D_\mu)\, T_6 + D_r T_3 - D_\mu T_4 + (r^2 D_r + r\mu D_\mu + 5)\, T_1, \end{aligned}\right\} \tag{100}$$

and

$$t_4 = T_3 + T_5.$$

To bring (99) into a gauge-invariant form, we must first express $\lambda_i \epsilon_{jlm} \lambda_l \xi_m$ and $\xi_i \epsilon_{jlm} \lambda_l \xi_m$ in terms of $\lambda_j \epsilon_{ilm} \lambda_l \xi_m$ and $\xi_j \epsilon_{ilm} \lambda_l \xi_m$ in accordance with the identities (39) and then replace the term in $\epsilon_{ijk} \lambda_k$ by the proper combination of the tensors $\lambda_j \epsilon_{ilm} \lambda_l \xi_m$ and $\xi_j \epsilon_{ilm} \lambda_l \xi_m$ (cf. equation (42)). In this manner we find that the defining scalars of T_{ij} are

$$\left.\begin{aligned} \tau_1 = t_1 - t_2 - r\mu t_3, \quad \tau_2 &= t_2 + D_\mu (t_4 + r\mu t_2 + r^2 t_3) \\ \text{and} \qquad \tau_3 &= t_3 + D_r (t_4 + r\mu t_2 + r^2 t_3). \end{aligned}\right\} \tag{101}$$

This completes our discussion of axisymmetric tensors and forms.

6. The fundamental correlation tensor

The correlation* between the instantaneous velocity components u_i and u'_j at two different points in a turbulent medium has been the subject of intensive study in recent years. Let Q_{ij} denote this tensor:

$$Q_{ij} = \overline{u_i u'_j}. \tag{102}$$

In virtue of the equation of continuity, this tensor is solenoidal in both its indices. Also, if the turbulence is homogeneous (cf. Batchelor 1946),

$$Q_{ij}(-\boldsymbol{\xi}) = Q_{ji}(+\boldsymbol{\xi}). \tag{103}$$

From these remarks it follows that in axisymmetric turbulence Q_{ij} can be derived from a skew tensor of the form (49) and that the coefficients A, B, C and D of Q_{ij} have the forms given in equations (50). Moreover, the condition (103) now implies that the defining scalars, Q_1 and Q_2, of Q_{ij} are both even functions of r and $r\mu$.

* We shall follow Batchelor in reserving the word 'correlation' for the mean of the products considered; however, we shall retain for the term 'correlation coefficient' the meaning customary in the theory of probability.

For small values of r the form of Q_{ij} can be deduced from the expansions

$$\left.\begin{aligned} Q_1 &= \alpha_{00}+r^2(\alpha_{02}+\alpha_{22}\mu^2)+r^4(\alpha_{04}+\alpha_{24}\mu^2+\alpha_{44}\mu^4)+\dots \\ \text{and} \quad Q_2 &= \beta_{00}+r^2(\beta_{02}+\beta_{22}\mu^2)+r^4(\beta_{04}+\beta_{24}\mu^2+\beta_{44}\mu^4)+\dots, \end{aligned}\right\} \tag{104}$$

valid near $r=0$. Using these expansions in the expressions for A, B, C and D given in equations (50), we find

$$\left.\begin{aligned} A &= 2(\alpha_{02}-\alpha_{22}+\beta_{02})+2r^2[(2\alpha_{04}-\alpha_{24}+2\beta_{04})+\mu^2(\alpha_{24}-6\alpha_{44}+\beta_{24})]+\dots, \\ B &= -(2\alpha_{00}+\beta_{00})+r^2[(-4\alpha_{02}+2\alpha_{22}-3\beta_{02})+\mu^2(2\beta_{02}-\beta_{22}-8\alpha_{22})]+\dots, \\ C &= \beta_{00}+r^2[(3\beta_{02}-2\alpha_{22})+\beta_{22}\mu^2]+\dots, \\ D &= 2r\mu[(2\alpha_{22}-\beta_{02})+r^2\{2(\alpha_{24}-\beta_{04})+\mu^2(8\alpha_{44}-\beta_{24})\}]+\dots. \end{aligned}\right\} \tag{105}$$

The corresponding expansion for Q_{ij}, as far as the second power of r, is

$$\begin{aligned} Q_{ij} = {} & 2(\alpha_{02}-\alpha_{22}+\beta_{02})\,\xi_i\xi_j \\ & +[-(2\alpha_{00}+\beta_{00})+r^2\{(-4\alpha_{02}+2\alpha_{22}-3\beta_{02})+\mu^2(2\beta_{02}-\beta_{22}-8\alpha_{22})\}]\,\delta_{ij} \\ & +[\beta_{00}+r^2(3\beta_{02}-2\alpha_{22}+\beta_{22}\mu^2)]\,\lambda_i\lambda_j \\ & +2r\mu(\lambda_i\xi_j+\xi_i\lambda_j)\,(2\alpha_{22}-\beta_{02}). \end{aligned} \tag{106}$$

From equation (106) we readily find that the behaviour of the correlations, for small separations, for the four situations shown in figure 1 are

$$\left.\begin{aligned} &(a) \quad \overline{u_{\parallel}(0,0)\,u_{\parallel}(x,0)} = -2\alpha_{00}-2(\alpha_{02}+\alpha_{22})\,x^2+\dots, \\ &(b) \quad \overline{u_{\parallel}(0,0)\,u_{\parallel}(0,y)} = -2\alpha_{00}-4\alpha_{02}y^2+\dots, \\ &(c) \quad \overline{u_{\perp}(0,0)\,u_{\perp}(x,0)} = -(2\alpha_{00}+\beta_{00})-(4\alpha_{02}+6\alpha_{22}+\beta_{02}+\beta_{22})\,x^2+\dots, \\ &(d) \quad \overline{u_{\perp}(0,0)\,u_{\perp}(0,y)} = -(2\alpha_{00}+\beta_{00})-(2\alpha_{02}+\beta_{02})\,y^2+\dots, \end{aligned}\right\} \tag{107}$$

where $\parallel$ and $\perp$ refer to directions parallel and perpendicular to the direction of the mean flow, $\boldsymbol{\lambda}$, and x and y denote separations parallel and perpendicular to $\boldsymbol{\lambda}$. The coefficients of r^2 and $r^2\mu^2$ in the expansion of the two defining scalars of Q_{ij} can therefore be determined by measuring the curvatures of the correlation curves at the origin, for the four situations shown in figure 1.

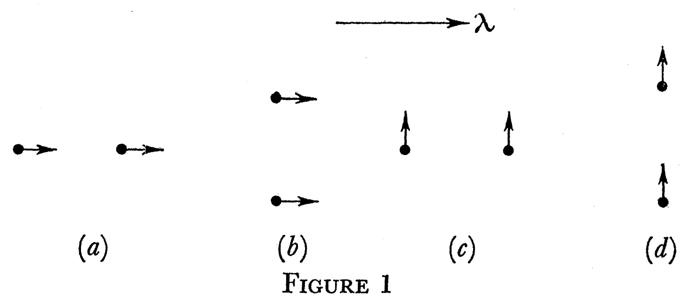

FIGURE 1

From equations (107) it follows, in particular, that

$$\overline{u_{\parallel}^2} = -2\alpha_{00} \quad \text{and} \quad \overline{u_{\perp}^2} = -(2\alpha_{00}+\beta_{00}). \tag{108}$$

Another quantity of interest is the trace, Q_{ii}, of the tensor Q_{ij}. Quite generally

$$Q_{ii} = r^2A+3B+C+2r\mu D, \tag{109}$$

and for A, B, C and D given by equations (50) this is

$$Q_{ii} = r^2(1-\mu^2)\,(D_{\mu\mu}Q_1 - D_r Q_2) - 2Q_2 - 2(r^2 D_r + 2r\mu D_\mu + 3)\,Q_1. \tag{110}$$

From this equation (or, more directly, from equations (105) and (109)) we find that for $r \to 0$

$$Q_{ii} = -2(3\alpha_{00} + \beta_{00}) + 2r^2[(\alpha_{22} - 5\alpha_{02} - 2\beta_{02}) + \mu^2(\beta_{02} - \beta_{22} - 8\alpha_{22})] + \ldots. \tag{111}$$

7. Scales of axisymmetric turbulence

In his paper on axisymmetric turbulence, Batchelor has defined four scales of axisymmetric turbulence by the formulae

$$\left.\begin{aligned} L_A &= (\overline{u_\parallel^2}\,\overline{u_\perp^2})^{-\frac{1}{2}}\int_0^\infty r^2 A\,dr, \quad L_B = (\overline{u_\parallel^2}\,\overline{u_\perp^2})^{-\frac{1}{2}}\int_0^\infty B\,dr, \\ L_D &= (\overline{u_\parallel^2}\,\overline{u_\perp^2})^{-\frac{1}{2}}\int_0^\infty rD\,dr, \quad L_C = (\overline{u_\parallel^2}\,\overline{u_\perp^2})^{-\frac{1}{2}}\int_0^\infty C\,dr, \end{aligned}\right\} \tag{112}$$

and has shown that L_A, L_B, L_C and L_D, as functions of μ, satisfy the differential equations

$$L_A - L_B - \mu\frac{dL_B}{d\mu} - 2\mu L_D + (1-\mu^2)\frac{dL_D}{d\mu} = 0$$

and

$$\frac{dL_B}{d\mu} - \mu L_C + (1-\mu^2)\frac{dL_C}{d\mu} + 2L_D = 0. \tag{113}$$

However, using the expressions for A, B, C and D given by equations (50) we find that we can evaluate the integrals defining L_A, L_B, L_C and L_D and express them explicitly in terms of the two functions

$$q_1(\mu) = -(\overline{u_\parallel^2}\,\overline{u_\perp^2})^{-\frac{1}{2}}\int_0^\infty Q_1\,dr$$

and

$$q_2(\mu) = -(\overline{u_\parallel^2}\,\overline{u_\perp^2})^{-\frac{1}{2}}\int_0^\infty Q_2\,dr. \tag{114}$$

Thus

$$\left.\begin{aligned} L_A &= q_1 + \mu q_1' + q_1'' + q_2 + \mu q_2', \\ L_B &= q_1 + \mu q_1' - (1-\mu^2)\,q_1'' + \mu^2 q_2 - \mu(1-\mu^2)\,q_2', \\ L_C &= q_1'' + \mu q_2' \end{aligned}\right.$$

and

$$L_D = -(q_1' + \mu q_1'') - \mu(q_2 + \mu q_2'), \tag{115}$$

where primes denote differentiation with respect to μ. It can be readily verified that the foregoing expressions for L_A, L_B, L_C and L_D satisfy Batchelor's differential equations (113). Indeed, the appearance of two arbitrary functions of μ in the solution (115) is in agreement with the fact that two differential equations govern four functions.

8. The dynamical equations

From the equations of motion we derive in the usual fashion that the equation governing the rate of change of Q_{ij} is (cf. von Kármán & Howarth 1938; and Batchelor 1946)

$$\frac{\partial Q_{ij}}{\partial t} = 2\nu\nabla^2 Q_{ij} + S_{ij}, \tag{116}$$

where ν is the coefficient of kinematic viscosity and

$$S_{ij} = \frac{\partial}{\partial \xi_k}(\overline{u_i u_k u'_j} - \overline{u_i u'_k u'_j}) + \frac{1}{\rho}\left(\frac{\overline{\partial p u'_j}}{\partial \xi_i} - \frac{\overline{\partial p' u_i}}{\partial \xi_j}\right). \tag{117}$$

In (117), the primes and the lack of primes distinguish the values of the quantities taken at x'_i and x_i, respectively; also ρ denotes the density and p the pressure.

We shall return to a detailed consideration of S_{ij} in § 10, but it is evident, meantime, that since both Q_{ij} and $\nabla^2 Q_{ij}$ are symmetrical in their indices and solenoidal, the tensor S_{ij} must also be symmetrical in its indices and solenoidal. Accordingly, S_{ij} must be derivable from a skew tensor of the form (49). Let S_1 and S_2 denote the defining scalars of S_{ij}.

Since the representation of the tensors and vectors we have adopted in terms of certain defining scalars is gauge-invariant and unique, we can directly pass from the tensor equation (116) to one between its defining scalars. Thus, remembering that the defining scalars of $\nabla^2 Q_{ij}$ are given by (61), we have

$$\frac{\partial Q_1}{\partial t} = 2\nu \Delta Q_1 + S_1 \tag{118}$$

and

$$\frac{\partial Q_2}{\partial t} = 2\nu(\Delta Q_2 + 2D_{\mu\mu} Q_1) + S_2, \tag{119}$$

where it may be recalled that

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2} - \frac{4\mu}{r^2}\frac{\partial}{\partial \mu} \quad \text{and} \quad D_{\mu\mu} = \frac{1}{r^2}\frac{\partial^2}{\partial \mu^2}. \tag{120}$$

Equations (118) and (119) are the fundamental equations in the theory of axisymmetric turbulence and replace the equation of von Kármán & Howarth in the theory of isotropic turbulence.

9. The rate of change of the mean squares of the velocity and the vorticity components: the viscous dissipation of energy

The equations governing the rate of change of the mean squares of the velocity and the vorticity components can be derived from equations (118) and (119) in the following manner:

First, we note that with the series expansions (104) for Q_1 and Q_2 we obtain for the defining scalars of $\nabla^2 Q_{ij}$ the expansions

$$\left.\begin{aligned} \Delta Q_1 &= (10\alpha_{02} + 2\alpha_{22}) + r^2[(28\alpha_{04} + 2\alpha_{24}) + \mu^2(12\alpha_{44} + 18\alpha_{24})] + \ldots, \\ \Delta Q_2 + 2D_{\mu\mu} Q_1 &= (10\beta_{02} + 2\beta_{22} + 4\alpha_{22}) + r^2[(28\beta_{04} + 2\beta_{24} + 4\alpha_{24}) \\ &\qquad + \mu^2(12\beta_{44} + 18\beta_{24} + 24\alpha_{44})] + \ldots. \end{aligned}\right\} \tag{121}$$

Also, let

$$\left.\begin{aligned} S_1 &= \eta_{00} + r^2(\eta_{02} + \eta_{22}\mu^2) + \ldots \\ \text{and} \quad S_2 &= \zeta_{00} + r^2(\zeta_{02} + \zeta_{22}\mu^2) + \ldots. \end{aligned}\right\} \tag{122}$$

Substituting these expansions for Q_1, Q_2, etc., in equations (118) and (119), we obtain the equations:

$$\left.\begin{aligned} \frac{d\alpha_{00}}{dt} &= 2\nu(10\alpha_{02} + 2\alpha_{22}) + \eta_{00}, \\ \frac{d\beta_{00}}{dt} &= 2\nu(10\beta_{02} + 2\beta_{22} + 4\alpha_{22}) + \zeta_{00} \end{aligned}\right\} \tag{123}$$

and

$$\left.\begin{aligned}\frac{d\alpha_{02}}{dt} &= 2\nu(28\alpha_{04}+2\alpha_{24})+\eta_{02},\\ \frac{d\alpha_{22}}{dt} &= 2\nu(12\alpha_{44}+18\alpha_{24})+\eta_{22},\\ \frac{d\beta_{02}}{dt} &= 2\nu(28\beta_{04}+2\beta_{24}+4\alpha_{24})+\zeta_{02},\\ \frac{d\beta_{22}}{dt} &= 2\nu(12\beta_{44}+18\beta_{24}+24\alpha_{44})+\zeta_{22}.\end{aligned}\right\} \tag{124}$$

The rate of change of the mean squares of the velocities parallel and perpendicular to $\boldsymbol{\lambda}$ can be obtained by combining equations (123) in accordance with (108). Thus,

$$\left.\begin{aligned}\frac{d\overline{u_\parallel^2}}{dt} &= -8\nu(5\alpha_{02}+\alpha_{22})-2\eta_{00}\\ \text{and}\qquad \frac{d\overline{u_\perp^2}}{dt} &= -4\nu(10\alpha_{02}+4\alpha_{22}+5\beta_{02}+\beta_{22})-(2\eta_{00}+\zeta_{00}).\end{aligned}\right\} \tag{125}$$

According to equations (125), the rate of dissipation of the energy is given by

$$\frac{d\overline{u^2}}{dt} = \frac{d}{dt}(\overline{u_\parallel^2}+2\overline{u_\perp^2}) = -8\nu(15\alpha_{02}+5\alpha_{22}+5\beta_{02}+\beta_{22})-2(3\eta_{00}+\zeta_{00}). \tag{126}$$

But (cf. equation (111))

$$[S_{ii}]_{r=0} = -2(3\eta_{00}+\zeta_{00}) \tag{127}$$

and this, as we shall see in § 10, vanishes. Hence

$$\frac{d\overline{u^2}}{dt} = -8\nu(15\alpha_{02}+5\alpha_{22}+5\beta_{02}+\beta_{22}). \tag{128}$$

In isotropic turbulence $\alpha_{22}=\beta_{02}=\beta_{22}=0$ and equation (128) reduces to the one first given by Taylor (1935, 1938). It can also be verified that equations (125) and (128) are in agreement with the ones given by Batchelor (1946) and derived by different methods.

The analogous equations for the rate of change of the mean squares of the vorticity components, ω_i, can be obtained by combining equations (124) appropriately; for it is well known in this theory (cf. von Kármán & Howarth 1938; also Batchelor 1946) that the mean squares of the vorticity components can be deduced from the coefficients in the expansion of the fundamental correlation tensor at the origin. In the present case, it follows from equation (106) that

$$\overline{\omega_i^2} = 4(5\alpha_{02}-3\alpha_{22}+4\beta_{02})+4(7\alpha_{22}-3\beta_{02})(1-\lambda_i^2). \tag{129}$$

Hence we have

$$\overline{\omega_\parallel^2} = 4(5\alpha_{02}-3\alpha_{22}+4\beta_{02}) \quad\text{and}\quad \overline{\omega_\perp^2} = 4(5\alpha_{02}+4\alpha_{22}+\beta_{02}). \tag{130}$$

10. The tensor S_{ij} and its defining scalars

As we have seen in § 8, the equation of motion for the fundamental correlation tensor Q_{ij} involves the tensor (equation (117))

$$S_{ij} = \frac{\partial}{\partial\xi_k}(\overline{u_i u_k u_j'}-\overline{u_j' u_k' u_i})+\frac{1}{\rho}\left(\frac{\partial\overline{pu_j'}}{\partial\xi_i}-\frac{\partial\overline{p'u_i}}{\partial\xi_j}\right). \tag{131}$$

Writing this in the form

$$S_{ij} = \frac{\partial}{\partial \xi_k} \overline{u_i u_k u_j'} + \frac{1}{\rho} \frac{\partial \overline{p u_j'}}{\partial \xi_i} - \left\{ \frac{\partial}{\partial \xi_k} \overline{u_j' u_k' u_i} + \frac{1}{\rho} \frac{\partial \overline{p' u_i}}{\partial \xi_j} \right\}, \tag{132}$$

we observe that the first two terms involve the correlations

$$\overline{u_i u_k u_j'} \quad \text{and} \quad \overline{p u_i'}/\rho. \tag{133}$$

The first of these represents the correlation, simultaneously, of two velocity components at $P(x_i)$ and one at $P'(x_i')$; it is clearly an axisymmetric tensor of order three, symmetrical in its first two indices and solenoidal in the third; it can accordingly be expressed in terms of six scalars, $T_1, \ldots, T_6$, in the form given by equation (95). The second of the two quantities (133) represents the correlation of the pressure at P and a velocity component at P'; it is clearly a solenoidal vector; it can accordingly be expressed in terms of a single scalar, ϖ, in the form (cf. equation (26))

$$P_i = \frac{1}{\rho} \overline{p u_i'} = -(r\mu D_r + D_\mu)\, \varpi \xi_i + (r^2 D_r + r\mu D_\mu + 2)\, \varpi \lambda_i. \tag{134}$$

The first two terms of S_{ij} are therefore

$$\frac{\partial}{\partial \xi_k} T_{ikj} + \frac{\partial P_j}{\partial \xi_i} = X_{ij} \quad \text{(say)}. \tag{135}$$

In the notation of §§ 4 and 5 (equations (72) and (97)) this is

$$X_{ij} = T_{ij} + P_{ij}. \tag{136}$$

Both of these tensors which make up X_{ij} are solenoidal in j, and their defining scalars have been given in equations (77) and (101). The defining scalars of X_{ij} are therefore

$$\left.\begin{aligned} \sigma_1 &= t_1 - t_2 - r\mu t_3 - (r\mu D_r + D_\mu)\, \varpi, \\ \sigma_2 &= t_2 + D_\mu(t_4 + r\mu t_2 + r^2 t_3) + D_\mu (r^2 D_r + r\mu D_\mu + 2)\, \varpi, \\ \sigma_3 &= t_3 + D_r(t_4 + r\mu t_2 + r^2 t_3) + D_r (r^2 D_r + r\mu D_\mu + 2)\, \varpi. \end{aligned}\right\} \tag{137}$$

An examination of these equations shows that t_2, t_3, t_4 and ϖ occur only in the combinations $t_2 + D_\mu \varpi$, $t_3 + D_r \varpi$ and $t_4 + \varpi$; and this, according to equations (100), means that T_3 and ϖ occur only in the combination $(T_3 + \varpi)$. Hence, with the definitions (cf. equations (100))

$$\left.\begin{aligned} s_1 &= (r^2 D_r + r\mu D_\mu + 5)\, T_4 + (r\mu D_r + D_\mu)\, T_5, \\ s_2 &= (r\mu D_r + D_\mu)\, T_2 + D_\mu (T_3 + \varpi) - D_\mu T_5 + (r^2 D_r + r\mu D_\mu + 5)\, T_6, \\ s_3 &= (r\mu D_r + D_\mu)\, T_6 + D_r (T_3 + \varpi) - D_\mu T_4 + (r^2 D_r + r\mu D_\mu + 5)\, T_1 \\ \text{and} \qquad s_4 &= (T_3 + \varpi) + T_5, \end{aligned}\right\} \tag{138}$$

we can write the defining scalars of X_{ij} in the forms (cf. equations (101))

$$\left.\begin{aligned} \sigma_1 = s_1 - s_2 - r\mu s_3, \quad \sigma_2 &= s_2 + D_\mu (s_4 + r\mu s_2 + r^2 s_3) \\ \text{and} \qquad \sigma_3 &= s_3 + D_r (s_4 + r\mu s_2 + r^2 s_3). \end{aligned}\right\} \tag{139}$$

Returning to equation (132), we observe that the terms in the curly brackets require the definition of the further correlations

$$\mathfrak{T}_{ijk} = -\overline{u_i' u_j' u_k} \quad \text{and} \quad \mathfrak{P}_i = -\overline{p' u_i}. \tag{140}$$

The tensor $\mathfrak{T}_{ijk}$, like T_{ijk}, is symmetrical in i and j and solenoidal in k. But the two tensors are not the same as in isotropic turbulence. However, by reflecting the vector configuration associated with T_{ijk} in the point P (see figure 2), we see that

$$T_{ijk}(-\boldsymbol{\lambda}) = \mathfrak{T}_{ijk}(+\boldsymbol{\lambda}). \tag{141}$$

Similarly,

$$P_i(-\boldsymbol{\lambda}) = \mathfrak{P}_i(+\boldsymbol{\lambda}). \tag{142}$$

Hence

$$-\frac{\partial}{\partial \xi_k}\overline{u_j' u_k' u_i} - \frac{1}{\rho}\frac{\partial \overline{p' u_i}}{\partial \xi_j} = \frac{\partial}{\partial \xi_k}\mathfrak{T}_{jki} + \frac{\partial \mathfrak{P}_i}{\partial \xi_j}$$

$$= \mathfrak{T}_{ji} + \mathfrak{P}_{ji} = T_{ji}(-\boldsymbol{\lambda}) + P_{ji}(-\boldsymbol{\lambda}) = X_{ji}(-\boldsymbol{\lambda}), \tag{143}$$

and

$$S_{ij} = X_{ij}(+\boldsymbol{\lambda}) + X_{ji}(-\boldsymbol{\lambda}). \tag{144}$$

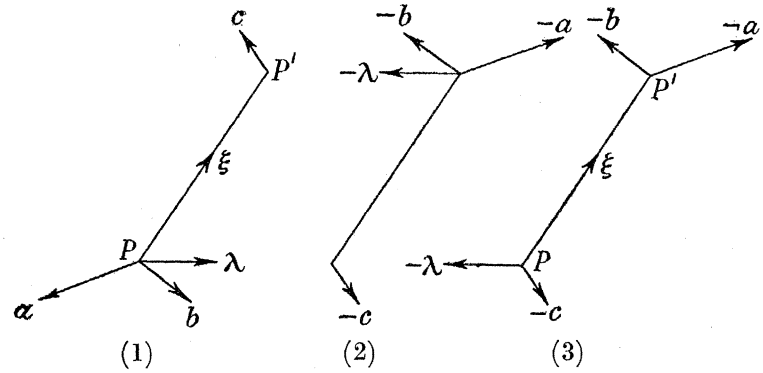

FIGURE 2. The vector configuration associated with $T_{ijk}a_i b_j c_k$ is represented by (1). The configuration (2) is obtained from (1) by reflexion in P; and this configuration is equivalent to (3) since, on account of homogeneity, we may choose any point as the origin P. This establishes the equality $[\overline{u_i u_j u_k'}]_\lambda = -[\overline{u_i' u_j' u_k}]_{-\lambda}$.

As we have already remarked in § 8, the equation of motion for Q_{ij} requires that S_{ij} be symmetrical and solenoidal in its indices. Now these conditions on S_{ij} actually require that X_{ij} be itself symmetrical in its indices. For, by writing X_{ij} in the form (34) and forming S_{ij} as prescribed, we can show that the solenoidal conditions, as expressed by equations (36) for the coefficients of X_{ij}, together with the similar equations for the coefficients of S_{ij} which are

$$\left.\begin{array}{c} A(r,+\mu)+A(r,-\mu), \quad B(r,+\mu)+B(r,-\mu), \quad C(r,+\mu)+C(r,-\mu), \\ D(r,+\mu)-E(r,-\mu) \quad \text{and} \quad E(r,+\mu)-D(r,-\mu), \end{array}\right\} \tag{145}$$

and the further condition that the last two of the coefficients (145) are equal require that $D = E$.

Now the condition that X_{ij} be symmetrical in its indices is (cf. equation (47))

$$\sigma_3 = D_\mu \sigma_1. \tag{146}$$

Substituting for σ_1 and σ_3 according to equations (139) in equation (146), we find that

$$(r^2 D_r + r\mu D_\mu + 4)\, s_3 + (r\mu D_r + D_\mu)\, s_2 = D_\mu s_1 - D_r s_4. \tag{147}$$

With equation (147) satisfied, the two defining scalars of X_{ij} are σ_1 and σ_2; and the defining scalars of S_{ij}, then, are

$$\left.\begin{array}{l} S_1 = \sigma_1(r,+\mu) + \sigma_1(r,-\mu) \\ S_2 = \sigma_2(r,+\mu) + \sigma_2(r,-\mu). \end{array}\right\} \tag{148}$$

and

The vanishing of the trace of S_{ij}, which was assumed in § 9 (equation (127)), follows from the fact that in any homogeneous turbulence T_{ij} must vanish at the origin and that, therefore,

$$[S_{ii}]_{r=0} = 2[P_{ii}]_{r=0}; \tag{149}$$

and that this last vanishes, follows directly from the representation of P_i in terms of ϖ.

It is worthy of notice here that according to the formulae we have given for the defining scalars of S_{ij} (cf. equations (122))

$$(S_1)_{r=0} = \eta_{00} = -2(D_\mu \varpi)_{r=0} \quad \text{and} \quad (S_2)_{r=0} = \zeta_{00} = +6(D_\mu \varpi)_{r=0}. \tag{150}$$

11. Concluding remarks

The developments of the preceding sections bring the formal theory of axisymmetric turbulence to the same degree of completeness that von Kármán, Howarth and Robertson brought the theory of isotropic turbulence. The principal result of this paper is, of course, the derivation of equations (118) and (119). It is evident that these equations must play in the theory of axisymmetric turbulence the same role which the equation of von Kármán & Howarth has played in the theory of isotropic turbulence. As to how far the present theory of axisymmetric turbulence will go towards clarifying the manner in which isotropic turbulence comes to prevail and towards accounting for the reappearance of anisotropy under certain conditions cannot be foretold without an investigation into the behaviour of the solutions of equations (118) and (119). It is the intention of the writer to return to these matters in a later communication.

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